Inequality 5

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Home Work :- Let a, b, c be positive real numbers such that, $abc = 1$

Prove that,

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1$$

Solution :-

WLOG $a \geq b \geq c$

$$(a^3-b^3)(a^2-b^2) \geq 0$$

$$\Rightarrow a^5 - b^3a^2 - a^3b^2 + b^5 \geq 0$$

$$\Rightarrow a^5+b^5 \geq a^2b^2(a+b) \Rightarrow \frac{1}{a^5+b^5} \leq \frac{1}{a^2b^2(a+b)}$$

$$\frac{ab}{a^5+b^5+ab} \leq \frac{ab}{a^2b^2(a+b)+ab} = \frac{abc^2}{a^2b^2c^2(a+b)+abc^2} = \frac{c}{a+b+c}$$

Similarly, $\frac{bc}{b^5+c^5+bc} \leq \frac{b}{a+b+c}$ and $\frac{ca}{c^5+a^5+ca} = \frac{b}{a+b+c}$

So, LHS ≤ 1

Q) $A_1 = (a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2$ $\quad a<b<c<d \in \mathbb{R}$

$A_2 = (a-c)^2+(c-b)^2+(b-d)^2+(d-a)^2$

$A_3 = (a-b)^2+(b-d)^2+(d-c)^2+(c-a)^2$

Prove that,

$$A_2 > A_1 > A_3$$

A :- $A_1 = 2(a^2+b^2+c^2+d^2) - 2(ab+bc+cd+da)$

Ans:- $A_1 = 2(a^2+b^2+c^2+d^2) - 2(ab+bc+cd+da)$

$A_2 = 2(a^2+b^2+c^2+d^2) - 2(ac+bc+bd+da)$

$$A_2 - A_1 = 2(ab+bc+cd+da) - 2(ac+bc+bd+da)$$

$$= 2(ab - ac + cd - bd)$$

$$= 2(a(b-c)+d(c-b))$$

$$= 2(b-c)(a-d) = 2\underset{>0}{(c-b)}\underset{>0}{(d-a)} > 0$$

$A_2 > A_1$

$$A_3 = 2(a^2+b^2+c^2+d^2) - 2(ab+bd+dc+ac)$$

$$A_1 - A_3 = 2(ab+bd+dc+ac) - 2(ab+bc+cd+da)$$

$$= 2(bd - bc + ac - da)$$

$$= 2(b(d-c)+a(c-d))$$

$$= 2\underset{>0}{(d-c)}\underset{>0}{(b-a)} > 0$$

$A_1 > A_3$

$\Rightarrow \quad A_2 > A_1 > A_3$

Roots of a quadratic function:-

$$ax^2+bx+c=0 \qquad a \neq 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow \quad x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{\ }\right)^2 = -\frac{c}{\ } + \left(\frac{b}{\ }\right)^2 = \frac{b^2-4ac}{\ }$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac > 0 \Rightarrow$ Two unique roots exists

$b^2 - 4ac = 0 \Rightarrow$ Only one real root exists

$b^2 - 4a < 0 \Rightarrow$ No real roots.

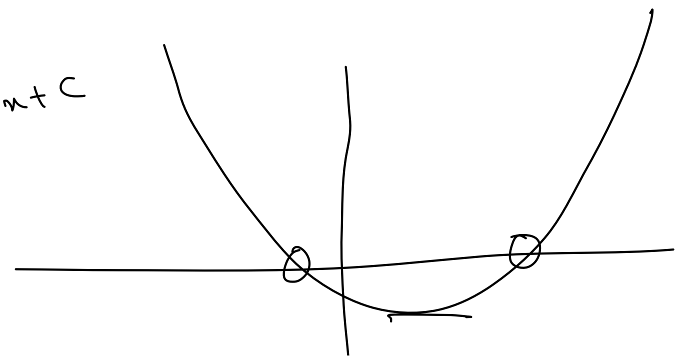
But we have two roots in complex domain which are conjugate of each other.

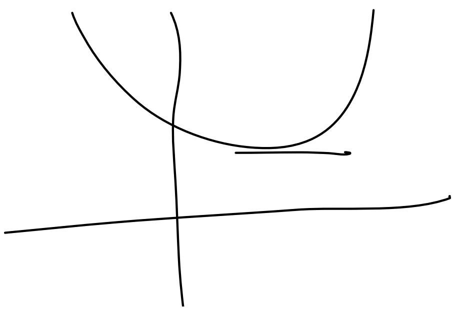
Q) Suppose the polynomial $ax^2 + bx + c$ satisfies the following: $a > 0$, $a + b + c \geq 0$, $a - b + c \geq 0$, $a - c \geq 0$, $b^2 - 4ac \geq 0$. Prove that the roots are real and belong to $-1 \leq x \leq 1$.

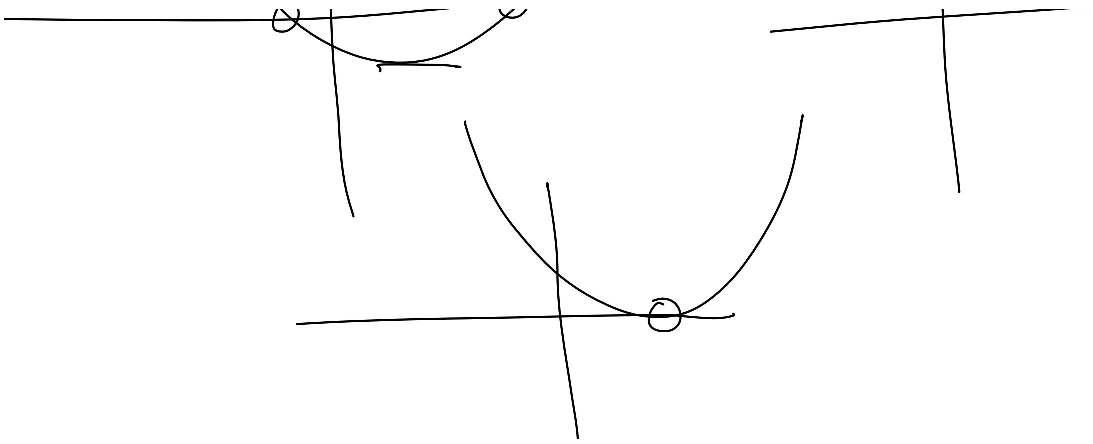
Ans:- HomeWork

$ax^2 + bx + c$

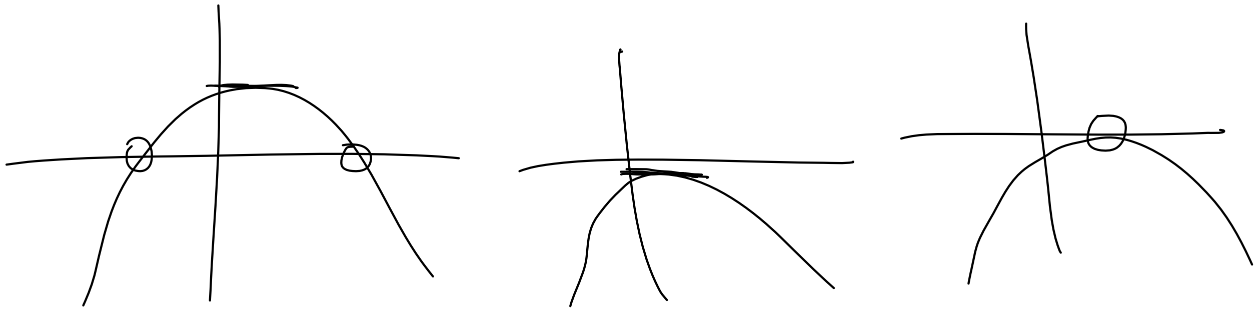
$a > 0$







$a<0$



$$\left(x=-\frac{b}{2a},\ y=\frac{-b^2}{4a}+c\right)$$

Minimum value is only at one point for $a>0$

Maximum " " " " " " $a<0$

$a>0$, $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$

$\left(x+\frac{b}{2a}\right)^2 \geq 0$ So minimum at $\boxed{x=-\frac{b}{2a}}$

So minimum value is $a\left(\frac{b}{2a}\right)^2 - b\left(\frac{b}{2a}\right)+c$

$$= \frac{b^2}{4a}-\frac{b^2}{2a}+c = \boxed{-\frac{b^2}{4a}+c}$$

$a<0$, $ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$

as $a<0$, $\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}$ must be minimum as $a<0$, $\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$ must be minimum

i.e, when $\boxed{x = -\frac{b}{2a}}$ ← Maximum at

Maximum value is $\boxed{\frac{-b^2}{4a}+c}$

Home Work

Q) If a, b, c are positive numbers, prove that it is not possible for the inequalities $a(1-b) > \frac{1}{4}$, $b(1-c) > \frac{1}{4}$, $c(1-a) > \frac{1}{4}$ to hold simultaneously

Home Work

Q) Given $a, b, c > 0$, it is possible to construct a triangle of side lengths a, b, c iff $pa^2 + qb^2 > pqc^2$ for any p, q with $p+q=1$